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Inference and Optimal Design of Accelerated Life Test using Geometric Process for Generalized Half-Logistic Distribution under Progressive Type-II Censoring

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Abstract

In this paper, the geometric process model is used for analyzing constant stress accelerated life testing. The generalized half logistic lifetime distribution is considered under progressive type-II censoring. Statistical inference is developed on the basis of maximum likelihood approach for estimating the unknown parameters and getting both the asymptotic and bootstrap confidence intervals. Besides, the predictive values of the reliability function under usual conditions are found. Moreover, the method of finding the optimal value of the ratio of the geometric process is presented. Finally, a simulation study is presented to illustrate the proposed procedures and to evaluate the performance of the geometric process model.

Key words and phrases: *Accelerated life test, Geometric process, Generalized Half-Logistic distribution, Progressive type-II censoring, Maximum likelihood estimation, Fisher information matrix, Bootstrap confidence intervals, Optimum test plan.*

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1. Introduction

The modern advancement in manufacturing electronic and mechanic products may contribute to delay failures. To overcome this difficulty, accelerated life testing (ALT) is applied by exposing test units to severe conditions (stresses) in order to get failure data rapidly. Then, the results will be extrapolated to the normal use conditions through a certain function that relates one of the parameters of the lifetime distribution with the stress(es). For more details on statistical inference of costant stress ALT, see for example: Nelson (1990), Abdel-Ghaly et al. (1998), Escobar and Meeker (2006), Aly and Bleed (2013), Abdel-Ghaly et al. (2016a), and Abdel-Ghaly et al. (2016b).

Recently, Geometric process (GP) became a beetter choice for analyzing ALT because of its simple nature, since it does not require a function to reparametrize the original parameters with a stress (s). Lam(1988 a,b) first introduced the GP. He studied a new repair replacement model for a deteriorating system. Iin this model, both the successive survival times of the system, and the consecutive repair times after failure, form a GP which is stochastically non-increasing for the first and non-decreasing for the second. Lam and Chan (1998) studied the statistical inference for the GP assuming lognormal distribution. Huang (2011) was the first who applied the GP model for analyzing ALT. He stated that since in ALT, lifetimes would be stochastically decreasing with respect to increasing stress level, then the GP model may be a natural approach to study such a problem. He considered constant stress accelerated life test (CSALT) and used the exponential lifetime distribution under each of complete sampling, type-I and type-II censoring. He used maximum likelihood estimation (MLE) method and derived the asymptotic confidence Interval (CI) for estimators and compared them with the parametric-bootstrap confidence intervals (CI). Zhou et al. (2012) used GP for implementing CSALT based on progressive type-I hybrid censoring. Considering, Rayligh lifetime distribution, they got the ML estimators and bootstrap CI.

Using different lifetime distributions and assuming complete data, several authors applied GP for estimating CSALT. For example; Kamal et al. (2012), Kamal et al. (2013), and Anwar et al. (2013) used Weibull, Pareto, and Marshall-Olkin extended exponential distributions, respectively. While, Rahman et al. (2016) and Ullah et al. (2017) used generalized exponential and generalized Rayleigh Distributions, respectively. On the other hand, under type-I censoring, Kamal (2013), Saxena et al. (2013) and Anwar et al. (2014) used Weibull, log-logistic, and Marshall-Olkin extended exponential distributions, respectively.

Type-I and type-II censoring schemes are the most commonly used in reliability engineering experiments. In the experiment, some units fail and the other are censored by assuming an ending time of the experiment or after a prespecified number of failures, using type-I and type-II, respectively. There is no possibility to remove any unit while the experiment is running. While, progressive censoring scheme allows some pre-determined units to be removed from the test at certain points of time before the end of the experiment. Using progressive censoring, the items are removed from life test throughout the duration of the test. This means that at various stages of the test, some of the survivors are withdrawn from further observations. Sample units which remain after each sample stage of censoring continue to be observed until failure or a subsequent stage of censoring.

In this paper, the generalized half-logistic (GHL) distribution is considered in ALT to represent the lifetime. For other applications different from ALT, several authors used GHL. For example; Arora et al. (2010) obtained maximum likelihood estimators (MLE) and the asymptotic variance of GHL under progressive type-I censoring with changing failure rates. Under progressive Type-II censoring, Kim et al. (2011) derived approximated profile MLE of the s-cale parameter of GHL. Seo et al. (2013) derived ML estimators of the unknown parameters of the generalized half logistic (GHL) distribution under type-II hybrid censoring. They also obtained approximate CI using asymptotic variance-covariance matrix based on ML approach. Balakrishnan and Saleh (2013) established several recurrence relations for the single and product moments of progressively type-II right-censored order statistics from a generalized half-logistic distribution. Rao and Rao (2014) used half-logistic distribution to analyze acceptance samples. Mohan et al. (2016) estimated the scale and shape parameters of GHL using the median ranks method.

It is advisable to have a plan that helps in accurately estimating the reliability at usual conditions. Before starting an ALT, a test plan needs to be developed to obtain appropriate and sufficient information in order to accurately estimate the reliability performance. Different approaches have been developed for planning experiments on the basis of covariance matrix, or equivalently, on the basis of Fisher Information (FI) matrix via the optimization of certain measurements of these matrices. In this paper, we will use both A optimality criterion.

This paper dels with applying GP to analyze CSALT using GHL distribution under progressive type-II censoring. While, all previous studies dealt with estimating the parameters only, this study is interested in not only the estimation but also the design of the experiment by getting the optimal value of the ratio of GP. The paper is organized as follows. In section 2, the used model is explained. Section 3 derives the ML estimators of the model parameters with their FI matrix, and then the predicted values of the relibility function is obtained. Both approximate and bootstrap CI are found in section 4. Optimum test plan is developed in section 5. Finally, the simulation studies needed for illustrating the theoretical results are presented in section 6. We concluded the paper in section 7.

2. The Model

In this section, the GHL distribution, concepts of GP and its application on ALT, progressive type-II censoring and the assumptions of the experiment are presented.

2.1 Generalized Half Logistic Distribution

The probability density function (pdf) of the generalized half-logistic distribution is given by (Seo et al. (2013))

$$f(x) = \frac{\beta}{\sigma}(\frac{2e^{-\frac{x}{\sigma}}}{1+e^{-\frac{x}{\sigma}}})^{\beta}\frac{1}{1+e^{-\frac{x}{\sigma}}}, \quad x>0, \quad \sigma,\beta>0. \tag{2.1}$$

The reliability function takes the form

$$R(x) = (\frac{2e^{-\frac{x}{\sigma}}}{1+e^{-\frac{x}{\sigma}}})^{\beta}. \tag{2.2}$$

and the corresponding hazard rate is given by

$$h(x) = \frac{\beta}{\sigma(1+e^{-\frac{x}{\sigma}})}. \tag{2.3}$$

This distribution has increasing hazard rate which is suitable for the concept of ALT.

2.2 Geometric Process

Lam (1988a,b) introduced the following definition for the geometric process.

Definition

For a sequence of non-negative random variables $X_l, l = 1, 2, ...$. If they are independent and the distribution function of X_l is given by $F(\lambda^{l-1}x)$ for $l = 1, 2, ...$, then $X_l, l = 1, 2, ...$ is called a geometric process(GP), where $\lambda > 0$ is the ratio of the GP.

A GP is (stochastically) increasing if the ratio $0 < \lambda \leq 1$; it is (stochastically) decreasing if the ratio $\lambda > 1$. A GP will become a renewal process if the ratio $\lambda = 1$. Therefore, GP is a simple monotone process and is a generalization of the renewal process. Thus, the probability function of x_l, could be written as

$$f_{X_l}(x) = \lambda^l f_{X_0}(\lambda^l x), l = 1, 2, ... \tag{2.4}$$

Theorem

When the underlying lifetime distribution is GHL, and the stress level in an ALT is increasing with a constant, i.e. $V_{k+1} - V_k = \triangle V, k = 1, 2, ..., S - 1$, then $[X_k, k = 0, 1, 2, ..., S]$ forms a geometric process, or log linear and GP models are equivalent when the stress increases arithmetically.

Proof

Assuming that the life characteristic σ of the product at any constant stress level V_k, is a log-linear function of the stress (Nelson (1990)) in the form:

$$\log(\sigma_k) = (a + bV_k),$$

where a and b are unknown parameters depending on the nature of the product under test. When $k = 0$, the above equation depicts the relationship at usual stress level, V_0. Then,

$$\log(\frac{\sigma_{k+1}}{\sigma_k}) = (bV_{k+1} - bV_k) = b\triangle V, or \frac{\sigma_{k+1}}{\sigma_k} = e^{b\triangle V}.$$

Now, assume that

$$\frac{\sigma_{k+1}}{\sigma_k} = \frac{1}{\lambda},$$

which shows that stress levels increases arithmetically with a constant difference, then

$$\sigma_k = \frac{\sigma_{k-1}}{\lambda} = \frac{\sigma_{k-2}}{\lambda^2} = ... = \frac{\sigma}{\lambda^k}.$$

Substituting in the pdf of GHL in equation (2.1),

$$f_{X_k}(x) = \frac{\beta\lambda^k}{\sigma}(\frac{2e^{-\frac{x\lambda^k}{\sigma}}}{1 + e^{-\frac{x\lambda^k}{\sigma}}})^\beta \frac{1}{1 + e^{-\frac{x\lambda^k}{\sigma}}}, \tag{2.5}$$

which could be written in the form

$$f_{X_k}(x) = \lambda^k f_{X_0}(\lambda^k x), k = 0, 1, 2, ..., S \tag{2.6}$$

which is as the same as equation (2.4).

2.3 Progressive Type-II Censoring

Cohen (1963) was the first who introduced Progressive type-II censoring scheme. It may be defined as follows:

Definition
Suppose n units are placed on a life test experiment and $x_1, x_2, ..., x_n$ denote the lifetimes of these n units taken from a population with lifetime distribution function $F(x;\theta)$ and density function $f(x;\theta)$, where θ is an unknown parameter(s) of interest. A pre-specified number of failures, $m < n$ is determined, then it is assumed that at the time of the j^{th} failure, $R_j > 0$ survived units are randomly removed from the experiment, $j = 1, 2, ..., m$. It is assumed that the censoring scheme $(R_1, R_2, ..., R_m)$ is fixed prior to study, such that $m + \sum_{j=1}^{m} R_j = n$. Data arising from such censored life testing experiment are referred to as progressively type-II censored data.

Thus, the likelihood based on the observed sample, $x_{(1)} < x_{(2)} < ... < x_{(m)}$ (for convenience notation are denoted by $x_1 < x_2 < ... < x_m$) is given by (see Balakrishnan and Aggarwala(2000))

$$L(\theta \mid \underline{x}) = C \prod_{i=1}^{m} f(x_i)[1 - F(x_i;\theta)]^{R_i}, \tag{2.7}$$

where $C = n(n-1-R_1)...(n - \sum_{j=1}^{m-1}(R_j + 1))$.

Recently, progressive censoring has been considered by many authors, for example, Balakrishnan and Aggarwala (2000) developed several techniques for analyzing progressive censored data, and Balakrishnan (2007) made a review on progressive censoring. Balakrishnan and Cramer (2014) introduced different applications for progressive censoring. Singh et al. (2015a) estimated flexible Weibull extension distribution under progressive type-II censoring. Singh et al. (2015b) proposed Bayesian estimation for the exponentiated gamma distribution under progressive type-II censored samples.

In the field of ALT, several authors considered progressive censoring. For example, Aly (2008) dealt with step-stress accelerated life test (SSALT) in the case of progressive type-I censoring using grouped data. It is assumed that the lifetime follows the log-logistic distribution and number of units removed at each stress is random following binomial distribution. ML Estimation and optimal test plan are obtained. Zhu and Elsayed (2013) investigated the design of AL plans under progressive censoring when test units experience competing failure modes and are subjected to either single or multiple stress types. Mohie El-Din et al. (2016) considered an extension of the exponential distribution under progressive censoring when applying CSALT. They estimated the parameters using ML and Bayesian approaches.

2.4 Assumptions

We assume the following assumptions for the CSALT procedure

- A total of N units are divided into n_1, n_2, ... , n_S units where $\sum_{k=1}^{S} n_k = N$.
- There are S levels of high stress V_k, $k = 1, ..., S$ in the experiment, and V_0 is the stress under usual conditions, where $V_0 < V_1 < \ldots < V_S$.
- Each n_k, $k = 1, \ldots, S$ units in the experiment are run at a pre-specified constant stress V_k, $k = 1, ..., S$.

- It is assumed that the stress affected only on the scale parameter of the underlying distribution.

- The failure times $x_{ik},\ i=1,...,n_k$ and $k=1,...,S$ at stress levels $V_k,\ k=1,\ldots,S$ have the GHL distribution with probability density function

$$f(x_{ik},\lambda,\sigma,\beta)=\frac{\beta\lambda^k}{\sigma}(\frac{2e^{-\frac{x_{ik}\lambda^k}{\sigma}}}{1+e^{-\frac{x_{ik}\lambda^k}{\sigma}}})^{\beta}\frac{1}{1+e^{-\frac{x_{ik}\lambda^k}{\sigma}}}. \tag{2.8}$$

- At each stress level, there is an experiment with m_k failures and $\Sigma_{j=1}^{m_k}R_j$ removals. Without loss of generality, we can assume that $m_k=m$ and $n_k=n$.

3. Maximum Likelihood Estimation

In the case of GHL using GP under progressive type-II censoring, the likelihood has the form:

$$L=\prod_{k=1}^{S}C_k\prod_{i=1}^{m}\frac{\beta\lambda^k}{\sigma}\frac{(2e^{-\frac{x_{ik}\lambda^k}{\sigma}})^{\beta}}{(1+e^{-\frac{x_{ik}\lambda^k}{\sigma}})^{\beta+1}}[\frac{2e^{-\frac{x_{ik}\lambda^k}{\sigma}}}{1+e^{-\frac{x_{ik}\lambda^k}{\sigma}}}]^{\beta R_i},$$

where $C_k=C=n(n-1-R_1)...(n-\sum_{j=1}^{m-1}R_j-m+1)$.

Thus, the likelihood function could be written in the form

$$L=\prod_{k=1}^{S}C\prod_{i=1}^{m}\frac{\beta\lambda^k}{\sigma}\frac{(2e^{-\frac{x_{ik}\lambda^k}{\sigma}})^{\beta(R_i+1)}}{(1+e^{-\frac{x_{ik}\lambda^k}{\sigma}})^{\beta(R_i+1)+1}}. \tag{3.1}$$

Taking the logarithm

$$\ln L=S\ln C+Sm\ln\beta+\frac{mS(S+1)\ln\lambda}{2}-Sm\ln\sigma+$$
$$\sum_{k=1}^{S}\sum_{i=1}^{m}\beta(R_i+1)[\ln 2-\frac{x_{ik}\lambda^k}{\sigma}]-\sum_{k=1}^{S}\sum_{i=1}^{m}[\beta(R_i+1)+1][\ln(1+e^{-}\frac{x_{ik}\lambda^k}{\sigma})]. \tag{3.2}$$

The first derivatives of the log-likelihood function (3.2) with respect to the unknown parameters β, σ and λ are

$$\frac{\partial\ln L}{\partial\beta}=\frac{Sm}{\beta}+\sum_{k=1}^{S}\sum_{i=1}^{m}(R_i+1)\ln(\frac{2\delta_{ik}}{1+\delta_{ik}}),or$$
$$\hat{\beta}=Sm/\sum_{k=1}^{S}\sum_{i=1}^{m}(R_i+1)\ln(\frac{1+\delta_{ik}}{2\delta_{ik}}), \tag{3.3}$$

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{Sm}{\sigma} + \frac{1}{\sigma^2}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{x_{ik}\lambda^k[\beta(R_i+1)-\delta_{ik}]}{1+\delta_{ik}}, or$$

$$\hat{\sigma} = (\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{x_{ik}\lambda^k[\hat{\beta}(R_i+1)-\delta_{ik}]}{1+\delta_{ik}})/Sm, and \tag{3.4}$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{mS(S+1)}{2\lambda} - \frac{1}{\sigma}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{kx_{ik}\lambda^{k-1}[\beta(R_i+1)-\delta_{ik}]}{1+\delta_{ik}}, \tag{3.5}$$

where $\delta_{ik} = e^{-\frac{x_{ik}\lambda^k}{\sigma}}$.

Solving (3.3), (3.4) and (3.5) numerically using Math-Cade program, the MLE of β, σ and λ could be obtained as shown in Section (6).

The second partial derivatives of the log-likelihood function (3.2) are as follows

$$\frac{\partial^2 \ln L}{\partial \beta^2} = \frac{-Sm}{\beta^2},$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \sigma} = \frac{1}{\sigma^2}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{(R_i+1)x_{ik}\lambda^k}{1+\delta_{ik}},$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \lambda} = -\frac{1}{\sigma}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{k(R_i+1)x_{ik}\lambda^{k-1}}{1+\delta_{ik}},$$

$$\frac{\partial^2 \ln L}{\partial \sigma^2} = \frac{Sm}{\sigma^2} - \frac{1}{\sigma^4}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{x_{ik}^2\lambda^{2k}\delta_{ik}[\beta(R_i+1)+1]}{(1+\delta_{ik})^2} - \frac{2}{\sigma^3}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{kx_{ik}\lambda^k[\beta(R_i+1)-\delta_{ik}]}{1+\delta_{ik}},$$

$$\frac{\partial^2 \ln L}{\partial \sigma \partial \lambda} = \frac{1}{\sigma^3}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{kx_{ik}^2\lambda^{2k-1}\delta_{ik}[\beta(R_i+1)+1]}{(1+\delta_{ik})^2} + \frac{1}{\sigma^2}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{kx_{ik}\lambda^{k-1}[\beta(R_i+1)-\delta_{ik}]}{1+\delta_{ik}},$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = \frac{mS(S+1)}{2\lambda^2} - \frac{1}{\sigma^2}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{kx_{ik}\lambda^{k-2}}{(1+\delta_{ik})^2}(kx_{ik}\lambda^k\delta_{ik}[\beta(R_i+1)+1] + \sigma(k-1)(1+\delta_{ik})[\beta(R_i+1)-\delta_{ik}]).$$

Therefore, the elements of the FI matrix for the MLE can be obtained as the expectations of the negative of the second partial derivatives, i.e.,

$$F = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ & f_{22} & f_{23} \\ & & f_{33} \end{pmatrix} = -E \begin{pmatrix} \frac{\partial^2 lnL}{\partial\beta^2} & \frac{\partial^2 lnL}{\partial\beta\partial\sigma} & \frac{\partial^2 lnL}{\partial\beta\partial\lambda} \\ & \frac{\partial^2 lnL}{\partial\sigma^2} & \frac{\partial^2 lnL}{\partial\sigma\partial\lambda} \\ & & \frac{\partial^2 lnL}{\partial\lambda^2} \end{pmatrix}. \tag{3.6}$$

The asymptotic variance-covariance matrix for the MLE is defined as the inverse of the Fisher information matrix (3.6), i.e.,

$$\Sigma = \hat{F}^{-1}. \tag{3.7}$$

Prediction under usual stress level

Under the usual stress V_0, the predicted parameter σ_0 will be estimated by

$$\hat{\sigma_0} = (\sum_{i=1}^{m} \frac{x_{i0}[\hat{\beta}(R_i+1) - \delta_{i0}]}{1+\delta_{i0}})/Sm, and \tag{3.8}$$

the MLE of the reliability function at the lifetime x_0, is given by

$$\hat{R}_0(x_0) = (\frac{2e^{-\frac{x}{\hat{\sigma_0}}}}{1+e^{-\frac{x}{\hat{\sigma_0}}}})^{\hat{\beta}}. \tag{3.9}$$

4. Confidence Intervals

In this section, we construct the approximate confidence intervals (CIs) for the parameters based on the asymptotic distributions of the estimators, and also the CIs using the parametric bootstrap approach.

4.1 Approximate Confidence Intervals

As shown in section (3), the MLEs $\hat{\Theta}$ are non-linear functions of random quantities, which make it virtually impossible to find their exact marginal/joint distributions in order to construct CIs. It is known that as the sample size grows, statistical inference about the unknown parameters can be based on the asymptotic normality of the MLEs. Thus, the vector $\hat{\Theta}$ is assumed to be approximately distributed as a multivariate normal with mean vector Θ and variance-covariance matrix $I_n^{-1}(\Theta)$. For more details on approximate CI, see Han and Kundu (2015).

In this subsection, we construct the approximate CIs for Θ as follows:

Let $\Theta = [\beta, \sigma, \lambda]$ and let W be a function of Θ. Then

$$var(\hat{W}) = [\frac{\partial W(\Theta)}{\partial\Theta}]'\Sigma_{\hat{\Theta}}[\frac{\partial W(\Theta)}{\partial\Theta}],$$

where

$$\Sigma_{\hat{\Theta}} = \begin{pmatrix} var(\hat{\beta}) & cov(\hat{\beta},\hat{\sigma}) & cov(\hat{\beta},\hat{\lambda}) \\ & var(\hat{\sigma}) & cov(\hat{\sigma},\hat{\lambda}) \\ & & var(\hat{\lambda}) \end{pmatrix}.$$

Using the asymptotic theory of MLE, the CI of W is given by

$$\hat{W} \pm z_{1-\alpha/2}\sqrt{var(\hat{W})}. \tag{4.1}$$

4.2 Bootstrap Confidence Intervals

Since all the information we have about the population is contained in the sample, bootstrap methods treat the sample as if it were the population. (see Efron and Tibshirani (1993) for details).

Percentile Bootstrap Confidence Intervals

To construct a two sided $100(1-\alpha)$ percentile bootstrap (PB) CI for the parameter Θ (in our case, $\Theta = (\beta, \sigma, \lambda)$), the following procedure is applied:

1. Estimate the unknown parameter Θ by the MLE $\hat{\Theta}$.
2. Using Θ, generate a bootstrap sample $x^*_{ik}, i = 1, 2, ..., m, k = 1, 2, ..., S$, where $x^*_{ik} \sim$ GHL distribution.
3. Get the estimates $\hat{\Theta}^* = (\hat{\beta}^*, \hat{\sigma}^*, \hat{\lambda}^*)$ from the bootstrap sample.
4. Repeat steps 2 and 3 Q=1000 times. Then, we have Q estimates of Θ.
5. Order the bootstrap replications of $\hat{\Theta}^*$ such that $\hat{\Theta}^*_1 \leq \hat{\Theta}^*_2 \leq ... \leq \hat{\Theta}^*_Q$.
6. The lower and upper confidence bounds are the $Q\frac{\alpha}{2}$ and $Q(1-\frac{\alpha}{2})$ ordered elements, respectively. Then, the 100 percent PB-CI for Θ is given by $(\hat{\Theta}^*_{Q-\frac{\alpha}{2}}, \hat{\Theta}^*_{Q(1-\frac{\alpha}{2})})$.

T-Bootstrap Confidence Intervals

The t-bootstrap (TB) CI for Θ is given by the following procedure:

1. Repeat steps 1 and 2 in PB method.
2. Get the estimate $\hat{\Theta}^*$ and compute $var(\hat{\Theta}^*)$ (in our case, $var(\hat{\beta}^*), var(\hat{\sigma}^*), var(\hat{\lambda}^*)$) using the observed Fisher information matrix
3. Compute the statistic $T^*_Q = \frac{\hat{\Theta}^* - \hat{\Theta}}{\sqrt{var(\hat{\Theta}^*)}}$.
4. Repeat steps 2 and 3 Q times.
5. Order the bootstrap replications of T^* such that $T^*_1 \leq T^*_2 \leq ... \leq T^*_Q$.
6. The lower and upper critical values are the $Q\frac{\alpha}{2}^{th}$ and $Q(1-\frac{\alpha}{2})^{th}$ elements, respectively. These critical values are used instead of those of T-tables. Thus, the $(1-\alpha)100$ percent bootstrap-t interval for Θ will be in the form

   $[\hat{\Theta} + t^*_{Q-\frac{\alpha}{2}}\sqrt{var(\hat{\Theta})}, \hat{\Theta} + t^*_{Q(1-\frac{\alpha}{2})}\sqrt{var(\hat{\Theta})}]$.

Alternatively, we can estimate symmetric critical values by:

1. Repeat steps 1 till 4 in the above method.
2. Order the bootstrap replications such that $|t_1^*| \leq |t_2^*| \leq ... \leq |t_Q^*|$. In this case, the lower and upper critical values are $-|t^*_{Q(1-\alpha)}|$ and $|t^*_{Q(1-\alpha)}|$. Then, the $100(1-\alpha)$ percent bootstrap-p interval for Θ has the form
   $[\hat{\Theta} - |t^*_{Q(1-\alpha)}|\sqrt{var(\hat{\Theta})}, \hat{\Theta} + |t^*_{Q(1-\alpha)}|\sqrt{var(\hat{\Theta})}]$.

5. Optimum Test Plan

This section is devoted to give the idea of finding the optimal value of λ, the GP ratio. In the field of life testing, it is known that designing the test before running the experiment is very important. In our case when applying ALT using GP, it is better to find a reasonable value of λ in order to get accurate results. We will use A optimality criterion.

The A optimality criterion is also known as trace criterion. It maximizes the sum of the diagonal entries of FI. The trace of FI, could be written as:

$$A = f_{11} + f_{22} + f_{33}.$$

Thus,

$$\begin{aligned}
\frac{\partial A}{\partial \lambda} &= f_{11}^{'} + f_{22}^{'} + f_{33}^{'} \\
&= -\frac{1}{\sigma^5}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{kx_{ik}^3\lambda^{3k-1}\delta_{ik}(\delta_{ik}-1)[\beta(R_i+1)+1]}{(1+\delta_{ik})^3} \\
&- \frac{4}{\sigma^4}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{kx_{ik}^2\lambda^{2k-1}\delta_{ik}[\beta(R_i+1)+1]}{(1+\delta_{ik})^2} + \frac{2}{\sigma^3}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{kx_{ik}\lambda^{k-1}[\delta_{ik}-\beta(R_i+1)]^2}{(1+\delta_{ik})} \\
&+ \frac{mS(S+1)}{\lambda^3} - \frac{5}{\sigma^2}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{k^2(k-1)x_{ik}^2\lambda^{2k-3}\delta_{ik}[\beta(R_i+1)+1]}{(1+\delta_{ik})^2} \\
&+ \frac{1}{\sigma}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{k(k-1)(k-2)x_{ik}\lambda^{k-3}[\delta_{ik}-\beta(R_i+1)]}{(1+\delta_{ik})^2} \\
&+ \frac{1}{\sigma^3}\sum_{k=1}^{S}\sum_{i=1}^{m}\frac{k^3x_{ik}^3\lambda^{3k-3}\delta_{ik}(\delta_{ik}-1)[\beta(R_i+1)+1]}{(1+\delta_{ik})^3}.
\end{aligned}$$

The optimal value of λ, λ^* is found by solving the equation

$$\frac{\partial A}{\partial \lambda} = 0. \tag{5.1}$$

We can also get the optimum value of $R_i, i = 1, 2, ..., m$, as

$$R_i^*(x_{ik}) = \left(\frac{2e^{-\frac{x_{ik}\lambda^{*k}}{\sigma}}}{1+e^{-\frac{x_{ik}\lambda^{*k}}{\sigma}}}\right)^{\beta}. \tag{5.2}$$

6. Simulation Studies

This section presents the numerical solutions to obtain the ML estimates of the unknown parameters β, σ, and λ, Moreover, their mean squared errors (MSE), relative absolute biases (RAB), CI using each one of the three introduced methods: approximate CI, PB, and TB methods. In order to explore the effects of several experimental parameters on the performance of estimation, the initial sample size n was chosen to be 9, 15, and 30. progressively Type-II censored samples from generalized half logistic distribution with different parameters values were generated using the algorithm described in Balakrishnan and Aggarwala (2000)and Balakrishnan (2007). Progressive Type II censoring schemes are chosen arbitrary as shown in Table 1.

The numerical analysis is performed based on 1000 Monte Carlo simulations with $B = 1000$ bootstrap replications. The actual coverage probabilities of the 95 percentage intervals for each model parameter were determined empirically as well as the relative absolute bias (RAB), and MSE associated with each estimator. The results are presented in Tables(1) and (2) along with the lower (L) and upper (U) bounds of the confidence intervals obtained using the above three methods.

It can be seen from the tabulated values that the estimates are slightly biased and that the MSE is very small and sometimes equals zero. Note that the estimates are quite stable and, more importantly, are close to the true values for the sample sizes considered. We also notice that the coverage probabilities of the approximate CI, PB and TB methods, are very high and in general greater than 0.93.

Tables (3) and (4) present the predicted values of the parameter σ of the GHL distribution and its reliability function using different sample sizes, and censoring schemes. It is seen that as the point of time, x_0 increses, the predicter reliability function, $\hat{R}(x_0)$ decreases. Sometimes, it reaches to zero.

7. Conclusion

In this paper, we have considered the estimation process of accelerated life test using geometric process for the generalized half logistic distribution under progressive type-II censoring. Point and interval estimations of the model parameters were obtained using the maximum likelihood approach. Prediction of the reliability function is found under the usual stress. Moreover, the idea of designing the test by searching for the optimal value of the ratio of geometric process is discussed. We have then conducted a simulation study to assess the performance of all these procedures. In the case of moderate to large sizes, the estimators give relatively accurate estimation of the parameters and appropriate coverage probabilities. We can say that simulation results indicate that the proposed geomeric process model works well using the generalized half logistic distribution under progressive type-II censoring. Thus, the geometric process is a good alternative to the log linear model that relates a certain parameter of the lifetime distribution with the high stress.

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Table 1 : The Coverage Probability (Cov. Prob.) of Approximate 95% C.I, RAB, MSE, and MLEs for Different Censoring Schemes

N	m	$Scheme$	$Parameter$	$Cov. Prob.$	$Approx.CI$ L-B	RAB	MSE	MLE
30	10	(2,0,0,2,0,0,0,2,0,14)	λ	0.9410	0.0038-0.0149	0.7580	0.0014	0.0121
			β	0.9509	0.0013-0.0064	0.9482	0.0090	0.0052
			σ	0.9444	0.0004-0.0007	0.945	0.0001	0.0005
		(0,0,1,1,0,0,1,1,1,15)	λ	0.9447	0.0231-0.0850	0.4426	0.0005	0.0721
			β	0.9426	0.0039-0.0193	0.815	0.0066	0.0185
			σ	0.9488	0.0017-0.0037	0.7283	0.0001	0.0027
		(5,0,5,0,0,0,0,5,0,5)	λ	0.9419	0.0212-0.0762	0.3269	0.0003	0.0663
			β	0.9427	0.0066-0.0266	0.7456	0.0056	0.0254
			σ	0.9461	0.0016-0.0034	0.7501	0.0001	0.0025
30	8	(8, 0, 0,1,0,4, 1, 8)	λ	0.9471	0.0033-0.0099	0.8084	0.0016	0.0096
			β	0.9452	0.0022-0.0075	0.9344	0.0087	0.0066
			σ	0.9397	0.0001-0.0002	0.9869	0.0001	0.0001
		(1,1,1,1,1,1,1,14)	λ	0.9477	0.0046-0.0151	0.7304	0.0013	0.0135
			β	0.9500	0.0013-0.0056	0.9592	0.0092	0.0041
			σ	0.9435	0.0001-0.0003	0.9816	0.0001	0.0002
15	5	(5,1,1,1,2)	λ	0.9506	0.0878-0.0922	0.8000	0.0016	0.0900
			β	0.9441	0.0038-0.0194	0.162	0.0000	0.0116
			σ	0.9406	0.0020-0.0031	0.786	0.0001	0.0021
		(1,1,1,1,6)	λ	0.9492	0.0173-0.0524	0.0168	0.0000	0.0492
			β	0.9396	0.0032-0.0099	0.5379	0.0000	0.0046
			σ	0.9496	0.0037-0.0114	0.2457	0.0000	0.0075
		(0,0,0,2,8)	λ	0.9494	0.0245-0.0656	0.1863	0.0001	0.0593
			β	0.9478	0.0035-0.0076	0.5885	0.0000	0.0041
			σ	0.9387	0.0027-0.0084	0.444	0.0000	0.0056
		(0,0,0,0,10)	λ	0.9459	0.0449-0.0995	0.9655	0.0023	0.0983
			β	0.9485	0.0010-0.0057	0.8293	0.0001	0.0017
			σ	0.9410	0.0024-0.0072	0.5188	0.0000	0.0048
9	3	(1,0,5)	λ	0.9467	0.0378-0.0567	0.0131	0.0000	0.0493
			β	0.9509	0.0031-0.0049	0.5996	0.0001	0.0040
			σ	0.9445	0.0051-0.0069	0.4487	0.0000	0.0055
		(0,0,6)	λ	0.9463	0.0607-0.0989	0.9321	0.0022	0.0966
			β	0.9430	0.0014-0.0034	0.8418	0.0001	0.0016
			σ	0.9440	0.0008-0.0040	0.7587	0.0001	0.0024

Table 2 : The Coverage Probability (Cov. Prob.) of Both P-Bootstrap CI and T-Bootstrap CI for Different Censoring Schemes

N	m	*Scheme*	*Parameter*	*Cov. Prob.*	*TBCI* L-B	*Cov. Prob.*	*PBCI* L-B
30	10	(2,0,0,2,0,0,0,2,0,14)	λ	0.9218	0.0069-0.0173	0.9548	0.0038-0.0149
			β	0.9417	0.0019-0.0085	0.9406	0.0013-0.0064
			σ	0.9584	0.0002-0.0013	0.9510	0.0004-0.0007
		(0,0,1,1,0,0,1,1,1,15)	λ	0.9274	0.0700-0.0991	0.9531	0.0231-0.0850
			β	0.9426	0.0132-0.0238	0.9305	0.0039-0.0193
			σ	0.9222	0.0020-0.0052	0.9511	0.0017-0.0037
		(5,0,5,0,0,0,0,5,0,5)	λ	0.9480	0.0563-0.0867	0.9429	0.0212-0.0762
			β	0.9561	0.0200-0.0345	0.9395	0.0066-0.0266
			σ	0.9259	0.0023-0.0043	0.9359	0.0016-0.0034
30	8	(8, 0, 0,1,0,4, 1, 8)	λ	0.9364	0.0056-0.0123	0.9341	0.0033-0.0099
			β	0.9292	0.0061-0.0114	0.9417	0.0022-0.0075
			σ	0.9517	0.0000-0.0015	0.9398	0.0001-0.0002
		(1,1,1,1,1,1,1,14)	λ	0.9497	0.0035-0.0166	0.9318	0.0046-0.0151
			β	0.9297	0.0006-0.0087	0.9438	0.0013-0.0056
			σ	0.9527	0.0000-0.0014	0.9403	0.0001-0.0003
15	5	(5,1,1,1,2)	λ	0.9262	0.0789-0.0111	0.9411	0.0878-0.0922
			β	0.9455	0.0021-0.0145	0.9432	0.0038-0.0194
			σ	0.9498	0.0011-0.0056	0.9409	0.0020-0.0031
		(1,1,1,1,6)	λ	0.9519	0.0292-0.0675	0.9386	0.0173-0.0524
			β	0.9572	0.0027-0.0120	0.9490	0.0032-0.0099
			σ	0.9385	0.0056-0.0095	0.9385	0.0037-0.0114
		(0,0,0,2,8)	λ	0.9512	0.0293-0.0711	0.9361	0.0245-0.0656
			β	0.9342	0.0036-0.0118	0.9400	0.0035-0.0076
			σ	0.9239	0.0034-0.0077	0.9471	0.0027-0.0084
		(0,0,0,0,10)	λ	0.9246	0.0068-0.0101	0.9384	0.0449-0.0995
			β	0.9425	0.0009-0.0104	0.9374	0.0010-0.0057
			σ	0.9455	0.0038-0.0058	0.9492	0.0024-0.0072
9	3	(1,0,5)	λ	0.9205	0.0293-0.0675	0.9453	0.0378-0.0567
			β	0.9486	0.0035-0.0165	0.9416	0.0031-0.0049
			σ	0.9294	0.0021-0.0090	0.9468	0.0051-0.0069
		(0,0,6)	λ	0.9393	0.0766-0.1002	0.9362	0.0607-0.0989
			β	0.9424	0.0012-0.0157	0.9379	0.0014-0.0034
			σ	0.9486	0.0004-0.0054	0.9398	0.0008-0.0040

Table 3: Estimated σ and $R(x)$ Under Usual Stress Using Different Schemes When $N = 30$

m	*Scheme*	$\hat{\sigma_0}$	x_0	$\hat{R}(x_0)$	m	*Scheme*	σ	x	$R(x)$
			0.01	0.9				0.01	0.62
			0.51	0.8				0.29	0.38
			0.56	0.71				0.44	0.23
			0.69	0.64				0.56	0.14
10	(2,0,0,2,0,0,0,2,0,14)	0.045	1.9	0.57	10	(0,0,1,1,0,0,1,1,1,15)	0.038	0.86	0.09
			2.62	0.51				1	0.05
			3.6	0.45				1.9	0.03
			3.82	0.4				3.6	0.02
			26.74	0.36				7.24	0.01
			32.12	0.32				26.74	0.01
			0.01	0.52				0.01	0.62
			0.29	0.26				0.29	0.39
			0.44	0.13				0.44	0.24
			1.9	0.07				0.51	0.15
10	(5,0,5,0,0,0,0,5,0,5)	0.038	2.62	0.03	8	(8, 0, 0,1,0,4, 1, 8)	0.014	0.69	0.09
			3.6	0.02				0.86	0.06
			3.82	0.01				3.82	0.04
			7.24	0				9.71	0.02
			9.71	0					
			26.74	0					
			0.01	0.74					
			0.29	0.55					
			0.51	0.41					
8	(1,1,1,1,1,1,1,14)	0.014	0.69	0.30					
			1	0.23					
			2.62	0.17					
			3.82	0.12					
			9.71	0.09					

Table 4: Estimated σ and $R(x)$ Under Usual Stress Using Different Schemes

N	m	*Scheme*	$\hat{\sigma_0}$	x_0	$\hat{R}(x_0)$	*Scheme*	σ	x	$R(x)$
15	5			0.05	0.62			0.05	0.97
				7.39	0.38			8.76	0.94
		(5,1,1,1,2)	0.028	1.9	0.23	(1,1,1,1,6)	0.153	34.75	0.92
				3.6	0.14			66.52	0.89
				7.24	0.09			108.9	0.86
15	5			0.05	0.96			0.05	0.97
				4.44	0.92			4.44	0.93
		(0,0,0,2,8)	0.094	8.76	0.88	(0,0,0,0,10)	0.049	8.76	0.9
				17.73	0.84			17.73	0.87
				66.52	0.81			66.52	0.84
9	3			0.05	0.97			0.05	0.94
		(1,0,5)	0.112	17.73	0.93	(0,0,6)	0.025	4.44	0.88
				79.3	0.9			17.73	0.83